CLASSIFICATIONS OF PERTURBATIONS OF RIGID MANIFOLDS

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This project is situated at the interface between Cauchy-Riemann (CR) geometry and dynamical systems. CR geometry initially studies real submanifolds in a complex euclidean space which endows them with a partial complex structure. It is then natural to classify them by diffeomorphisms preserving the complex structure of the ambient space. It happens that is some cases, methods borrowed to Dynamical systems can be implemented and are quite effective in order to understand the "biholomorphic equivalence problem of a given class of CR manifolds" Let us be more precise: Consider germs of generic real-analytic submanifolds of $\mathbb{C}^N$ of codimension d, i.e. real submanifolds which in suitable coordinates $(z,w) \in \mathbb{C}^n_z \times C^d_w$ can be written as $\operatorname{Im} w = \varphi(z,\bar z,\operatorname{Re} w)$ for a (holomorphic in a neighbourhood of $0 \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d$) real-valued function φ. We would like to understand when two such real submanifolds M, M' are *biholomorphically equivalent*, that is: under which conditions does there exist a *biholomorphism* $H(z,w) = (f(z,w), g(z,w))$ such that $H(M) = M'$? The application of dynamical systems methods for this type of problem is for example seen in the normal form of Chern and Moser [3] for Levi-nondegenerate hypersurfaces.

Some initial invariants are readily introduced: By considering the structure of the commutators of CR and anti-CR vector fields, the notion of *finite commutator type* gives rise to the Bloom-Graham invariants (or Hörmander numbers) at a point $p \in M$. One obtains this invariant as the sequence (m_j, ℓ_j) of orders m_j at which the space of commutators of CR and anti CR vector fields of length m_j increases its dimension by $\ell_j > 0$. One can show that for a real submanifold $M \subset \mathbb{C}^N$ of (finite) Bloom-Graham type $((m_j,\ell_j))_{j=1,\dots,d}$ there exists a choice of holomorphic coordinates $(z,w) \in \mathbb{C}^n \times \mathbb{C}^\ell$, where $w = (w_1,\dots,w_\ell) \in \mathbb{C}^{\ell_1} \times \cdots \times \mathbb{C}^{\ell_d}$ such that M is given by equations $\operatorname{Im} w_j = P_j(z,\bar z,\operatorname{Re} w_1,\dots,\operatorname{Re} w_{j-1}) + \dots$ with P_j quasihomogeneous of degree m_j in the grading where z has weight 1 and w_j has weight m_j; see e.g. [1]

A couple of years ago, Beloshapka [2] has completed a study of the associated model surfaces, and characterized holomorphic nondegeneracy of them by an algebraic condition in the special case where the equations are of the form

$$\mathcal{I} : \operatorname{Im} w_j = P_j(z,\bar z), \quad j = 1,\dots,d$$

with P_j homogeneous of degree m_j; these are said to be *rigid*. A specially appealing case is if the Bloom-Graham type stays constant on $\mathcal{I}$ as this is equivalent to holomorphic homogeneity of $\mathcal{I}$. We propose to study this case first, and so we now consider a real analytic higher order perturbation $\mathcal{M}$: $\operatorname{Im} w_j = P_j(z,\bar z) + \Phi_j(z,\bar z,\operatorname{Re} w)$, $j = 1,\dots,l$ where $\Phi_j(z,\bar z, Rew)$ is of quasiorder greater than m_j.This situation shares some similarities with the one considered by the advisors B. Lamel and L. Stolovitch in [4], where we considered Levi-nondegenerate submanifolds of high codimension with a similar approach as the one suggested here. In that case, the P_j are quadratic polynomials.

There are some interesting aspects related to this problem which could be pursued in parallel to the dissertation problem; for example, one could ask the relationship with *rigid normal forms* (which keep the special automorphisms $w \mapsto w + r$, $r \in \mathbb{R}^d$, in this form) as in Stanton's work [5]. Another such aspect is the identification of real submanifolds with local symmetry algebras of maximal dimensions in special cases.

We also think about this problem as a test case for building general conditions for finding "Big Denominators" phenomena (see [6]) in conjugacy problems, i.e. an appropriate decay (wrt to order) of the estimates of solutions of the linearized conjugacy equations sufficient to infer holomorphic conjugacy to a normal form), and of course, it would be interesting to study the general (non-rigid) case as a follow-up. Summarizing, the thesis project has the following two complementary aspects:

CR Geometry Goal: Define a normal form for the perturbation of finite Bloom-Graham type submanifolds and find a condition under which the transformation to such a normal form converges.

Dynamical Systems Goal Identify simple conditions ensuring that a conjugacy problem has the Big Denominators property.

Necessary Prerequisites: It would be ideal if the successful applicant has some background in Several Complex Variables and/or Dynamical Systems. We believe that the specialized knowledge necessary to carry out the dissertation projectis accessible to most students with a thorough analysis background.

References

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