HIGHER ORDER MIXING IN PARABOLIC DYNAMICS

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The focus of ergodic theory is the study of measure-preserving dynamical systems which, despite their deterministic nature, display chaotic, "random-like" behaviours. One of the fundamental manifestations of chaos in measure-preserving dynamical systems is *mixing*[1]. In probabilistic language, mixing means that any two observables become asymptotically independent under the action of the dynamics, or, in other words, their correlations decay. More in general, a measure-preserving system is *k-mixing* if the multiple correlations between any k observables decay. Although it has been proven that several mixing systems are also mixing of all orders (i.e., k-mixing for all $k \geq 2$), the general question whether 2-mixing implies mixing of all orders remains a famous open problem in ergodic theory (the so-called *Rohlin Problem*).

Once that a given system is known to be mixing or k-mixing, it is of fundamental importance, especially for applications, to estimate *how fast* the correlations between smooth observables decay. This has been done successfully for several "highly chaotic" (*hyperbolic*) systems [3, 21] and for some *parabolic* ones [5]. The adjective *parabolic* refers to those dynamical systems which display an intermediate regime between ordered ergodic (elliptic) systems and highly chaotic (hyperbolic) ones. Important examples of such systems are smooth area-preserving flows on surfaces, unipotent flows on quotients of semisimple Lie groups, nilflows on nilmanifolds, and their smooth perturbations, in particular their time-changes. Parabolic flows are known to be typically mixing, but with a slow (namely, polynomial) rate, see, e.g., [1, 6, 7, 8, 9, 11, 12, 17, 20] and references therein for a complete panorama.

The aim of the present proposal for a PhD research project is to advance the current knowledge on mixing properties, in particular on higher order mixing, for smooth parabolic flows, both homogeneous and non-homogeneous. There have been several new developments in this direction in recent years, but the current state of the art is far from satisfactory. We believe that the open questions we describe in this proposal can be addressed by a PhD student and positive answers will attract attention from the scientific community working in parabolic dynamics. Also the PhD student will find sufficient know-how in the ergodic theory group (Davide Ravotti, Roland Zweimüller, in addition to the supervisor Bruin) to find a helpful soundboard.

Specific projects:

(1) The *horocycle flow* is arguably the prime example of a homogeneous parabolic flow. Sharp mixing estimates were established by Ratner in her seminal paper [17]. Mixing of all orders was proved by Marcus [15], and a polynomial upper bound for k-mixing, for all $k \geq 3$, follows from a result of Björklund, Einsiedler and Gorodnik [2], which holds in great generality. However, in this specific setting, one could aim at obtaining sharper estimates for higher order mixing. A possible approach to tackle this problem is to combine Ratner's strategy in [17] with either [2] or [15].

(2) The exponent governing the rate of decay of k-correlations in [2] is not explicit. It would be interesting to consider specific cases, e.g., for unipotent flows on quotients M of $\mathrm{SL}_n(\mathbb{R})$, and estimate it for given $k \geq 3$ (in terms of the geometry of M), or parabolic interval (Pomeau-Manneville) maps [4, 5].

(3) The study of non-homogeneous parabolic flows has received considerable attention in the last few years. In particular, researchers have tried to generalise the results known for unipotent flows to their smooth perturbations, in particular to their time-changes (see, among others, [1, 10, 11, 13, 16]). Compared to the unperturbed setting, much less is known on their quantitative mixing properties. It is now known that smooth time-changes of unipotent flows are polynomial mixing [11, 19]; however, going beyond 2-mixing appears to be a challenging problem. Proving a quantitative result for higher order mixing for these flows is the main goal of the present proposal and would definitely be an impressive achievement for a PhD candidate. To this end, there are several intermediate steps that are most likely within reach and are nonetheless interesting results worth of publication in good mathematical journals. By combining the techniques in [14] and [19], it should be possible to prove polynomial 3-mixing for smooth time-changes of general unipotent flows. Similarly, one can investigate 3-mixing rates for smooth area-preserving flows on surfaces, in the spirit of [18, 20].

[1] Sometimes referred to as *strong mixing* or *strong 2-mixing*.

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