Vienna School of Mathematics, outline of a PhD project on

Rare events - probabilistic laws from deterministic dynamics

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Areas: ergodic theory of dynamical systems & probability theory

Outline: It is well known that various apparently simple deterministic dynamical systems exhibit chaotic behaviour, meaning that a detailed prediction of their long-term behaviour is inherently impossible. Remarkably, though, ergodic theory sometimes enables us to still derive useful and *rigorous* statements about their asymptotics. These results are of a quantitative nature, describing, say, how often certain events will occur in the long run. Indeed, the famous Ergodic Theorem from the 1930ies does exactly that, and thus generalizes a fundamental result of probability, the law of large numbers, to a much wider class of processes which are not a priori given in probabilistic terms. Today we can go much further and study finer quantitative features of the processes generated by dynamical systems, analogously extending other probabilistic limit theorems (like the central limit theorem, invariance principle, large deviation principle etc), contributing to a rigorous quantitative theory for chaotic dynamical systems.

The present project focuses on *understanding at what times certain rare but important events occur in such systems*, by analysing the behaviour of hitting-time processes of small sets in the phase space, their limits (Poisson processes or more general Markov processes) as the size of the target sets tends to zero, and their relation to other dynamical and probabilistic features of the system.

What flavour of maths? While motivated by important real-world phenomena, we focus on the mathematical foundations, and hence on the abstract theory and the rigorous study of prototypical models. We wish to understand which dynamical mechanisms are responsible for the emergence of classical probabilistic laws in deterministic systems. Often the best starting point is some seemingly simple toy system, say a chaotic map T on an interval, like $Tx := 4x(1-x)$ on $[0, 1]$. For example, the waiting time for the first success in a sequence of Bernoulli trials (coin flips) converges to an exponential law as the success probability tends to zero. The same is true for the first time at which the orbit $(T^n x)_{n \geq 0}$ under T of a randomly selected point x hits a given small subinterval (in the limit, as the latter shrinks to a typical point). The case of Bernoulli sequences is really easy. But proving and understanding the corresponding result for T is a very different matter which requires us to tie together dynamical systems theory and probability. Finding out how to do this in the toy model, however, is already a crucial step towards a much broader understanding of the issue which can then be pursued on a more abstract level.

This project is embedded in a larger program involving another PhD student, a PostDoc researcher, and international cooperation partners. There is flexibility regarding the concrete problems the candidate chooses to work on. Specific tasks range from the analysis of concrete (classes of) dynamical systems (mostly involving real and functional analysis and specific techniques from dynamics), to probabilistic questions (often about limit distributions and limit processes), and to abstract structural ergodic theory.

Required background: This circle of topics lies at the interface between the ergodic theory of dynamical systems and probability theory. Accordingly, it is essential for candidates to have a strong background in at least one of these areas, and a robust working knowledge of underlying analytical theories (measure theory, real and functional analysis).